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Mathematics News Letter

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This journal is dedicated to mathematics in general, to the following causes in particular (1) the common problems of grade, high school and college mathematics teaching, (2) the disciplines of mathematics, (3) the promotion of Mathematical Association of America and National Council of Teachers of Mathematics projects.

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OUR EDITORIAL EXPANSION

With representatives of at least a half dozen of the Louisiana-Mississippi major institutions of learning participating in the editorial direction of the News Letter, there is almost certain promise of a wider and more effective service to the cause of mathematics and its teaching in the home area of the journal. Especially gratifying to its friends and supporters should be the fact that Louisiana and Mississippi are now about equally represented in its management. The increase of personnel in the group of those who are to be responsible for a continued growth and increased effectiveness of the Letter means that the rate of this growth and increase will be quickened, as centers of responsibility for the journal have been multiplied and distributed over a wider range.

No prophet can with certainty predict what the News Letter may *not* become, or, over how wide an area its influence may *not* be felt in the years to come. However this may be, it is gratifying to know that, if the need should be developed for an even larger body of editors

for the News Letter, that need could be supplied without difficulty, since within these two States are institutions of strength other than those presently represented, institutions in which are mathematicians of promise and ambition.—S. T. S.

MATHEMATICS IN THE RUTS -- AMUSING, OR PATHETIC ?

Once upon a time, we believe we heard an ex-college teacher remark that a certain some one else (college teacher also) had lost much of his erstwhile skill as a teacher because he had latterly been spending too much of his time doing advanced study at graduate institutions. Put otherwise, he had been learning too much about his subject.

If we assume that the level of subject matter is at least the college level and that the subject matter is mathematics such an opinion is nothing but fallacy. An even stronger statement can be made. If the sharp edge of skilled technique of mere teaching of mathematics must suffer dulling because the teacher's love of mathematics may have led him to acquire more and more mathematics, then unquestionably the "skilled" technique is little more than worthless. Skill in imparting the facts of any science cannot in the very nature of things be grounded in a knowledge that is not permitted to grow from year to year as a result of continued enthusiastic study. A teaching technique that is not accompanied by enthusiasm for the subject taught is as lacking in effectiveness as a last year's bird's nest, and is just as dead. And, to this may be added yet another proposition. Mathematical knowledge that does not grow steadily can only imply an instructor whose material and methods are monotonously alike from year to year, and where monotony dwells enthusiasm refuses to live, indeed cannot live. But the truth that CAPS everything is this: Just as life can spring only from life, enthusiasm for mathematics is

only generated by the mathematical enthusiasm of another. If the teacher has it NOT, how can it be communicated to the student?

Rounding out the thought, we are lead once more to a view-point we have frequently presented editorially. The solid foundation of all correct technique of teaching is the creation in the student mind (it matters not *how*) of an abiding INTEREST in the subject. When this has been done the process of SELF-INSTRUCTION will have been begun, and technique can be thrown out of the window.—S. T. S.

INTELLIGENCE GROUPING IN MATHEMATICS

Among the many perplexing problems of the mathematics teachers is the problem of the minimum requirements for students of low mathematical ability. Modern text book writers have been providing differentiated assigned material for the three intelligence groups, known generally as the x, y, z groups. These assignments have been designated as maximum, average and minimum and this mode of procedure has been a distinct improvement upon the old plan of a single assignment for the whole group, which did not provide for the very bright and rapid worker, nor for the slow worker of low mathematical ability.

Have the so-called minimum assignments been wholly satisfactory? Have all of your students been able to master the minimum assignments? Have you felt that the minimum assignments had really been organized to fit the needs of pupils with low skills?

No scientific study through classroom investigation has been made along these lines but a group of investigators are formulating a plan for making a study of materials to be used with slow groups and the News Letter is interested in promoting a study along these lines. If you are interested in this movement, let us hear from you. If you are carrying on any study relative to this, will you give us the results of your study?—D. M. F.

MOTIVATION IN ALGEBRA

By JEANNE VAUTRAIN
New Orleans, La.

A constructive imagination is one of the most important elements in our life today. It is the man with the active, vivid imagination who is achieving the apparently impossible. The Wright Brothers and their sister, who financed their first experience with the airplane, were possessed of large imagination. Lindbergh with the imaginative "We" kept up spirit and strength until Paris came in sight. The teacher who can get mathematics to stimulate the imagination will fire his pupils to accomplishments.

Some teachers start a class in algebra by spending the first few days in reviewing arithmetic. This is a serious mistake. The pupil comes with an appetite for a new subject and he is curious to know what the subject is about. The wise teacher will use this time to fire the imagination. The attention of the pupil may be called to some of the great problems of mathematics. Talk to him of the great bridges and engineering feats and show the important part mathematics has in their construction. Tell him how mathematics furnishes the means of measuring the distance from sun to moon, how we can find the height of a mountain on which man has never set foot.

Through historical sidelights, the pupil comes to look upon his work as that which was created by human beings possessing human interests. Ways of gaining interest in the subject of algebra which is the basis of all higher mathematics are (1) telling of the discovery of the planet Neptune by means of mathematics, (2) location of big guns in the recent war, (3) showing how mathematics measures the earth, weighs the stars, (4) it is the law, the order, the beauty of the universe. There are numerous interesting historical sketches which may be given at various times. As a class takes up a new topic, an entertaining bit of its history may be given; for instance, the transposing of terms in an equation. (The name algebra came from "Al-jabr w'al mugabalah" which were the names of the two operations that were considered necessary to perform before the mechanical rule could be applied.)

In addition to history pictures, the pupils may be given recreation by means of drill given in the form of games, baseball, relay, races, etc. The formation of algebraic magic squares will furnish lively practice. A recreational appeal to show the importance of the plus and minus before a radical sign follows:

$$
\begin{array}{lrl}
 & & 9+5 = 2\times 7 \\
\text{(multiply by} & 9-5) & 9^2-5^2=2\times 7\times 9-2\times 7\times 5 \\
 & \therefore & 9^2-2\times 9\times 7 = 5^2-2\times 5\times 7 \\
\text{(adding } 7^2) & & 9^2-2\times 9\times 9+7^2=5^2-2\times 5\times 7+7^2 \\
\text{(taking square root)} & & 9-7= \quad 5-7
\end{array}
$$

Hence $9=5$, a result due to the failure to consider signs in taking square roots. Young people like the trick element and will master the details when their importance is demonstrated through a "trick".

A mathematics atmosphere is a great aid in the classroom—a few magazines, books on history of mathematics, a few pictures, a graphical presentation of the progress of the class, clippings which require a knoweldge of algebra for their understanding, posters, etc. The establishment of a museum appeals to the instinct to collect and to construct. If the pupil is on the lookout for material, he will soon learn to see mathematics in all life about him. He comes to realize that not only has mathematics a relation to life but it is life.

Organizing a club will vitalize the subject. The pupils will get an opportunity to see the subject from a different point of view. They can get more of the practical application and they get the "fun side" of the work. The club might entertain at a minstrel show where the X, Y, Z minstrels give mathematics jokes, songs, games. At an out-of-doors meeting, the members may do field problems requiring an algebraic solution. The officers may have titles from algebraic terminology and when having contests, the opponents may be named negatives, positives, etc. Various committees may report on history, relation of algebra to vocations (outside speakers may be invited), solving difficult problems, acquiring a mathematics library, preparing monthly contests, preparing roll of honor, etc.

The true test of teaching is in the thought which it provokes. A constructive imagination is the one thing which an individual must possess if he is to rise above those who imitate or work by rule of thumb. A constructive imagination, or live interest, will make of him a leader.

"THE INVESTIGATIVE METHOD vs. THE TRADITIONAL METHOD OF STUDYING PLANE GEOMETRY"

By **LEROY S. MILLER**
Louisiana Normal College

Tradition! What a mighty sovereign over the conduct of human beings. We move and act according to the dictates of tradition. If a thing has been done in a particular way for decades and generations, then tradition or custom demands that we follow the well-beaten path.

It is easy to follow in the path of tradition because it is the path of least resistance. The pathway is smoother, the way is well marked, and at every mile post there is a sign that points in the right direction to follow. On the other hand, it requires courage and perseverance for man to cut new trails and build new roads. To stand out as an exponent of things contrariwise to custom, to open new fields and set forth in the world of experimentation to find out what is best without regard to what these findings may be, to search for truth and let truth speak for itself is the test of genuine investigation and research.

Quite a number of educators have been blazing the way and advocating a departure from tradition in the study of Plane Geometry. How can we justify or condemn anything they may say or recommend as being the better method of study? Would it not be better to find out WHY we study Plane Geometry and then determine if the method suggested by these educators accomplish that aim?

Why should Geometry be taught in our schools? The two principal reasons usually given are for (1) its practical value, and (2) the culture it imparts. Needless to dwell upon the practical value of mathematics. All the industrial progress of the world, modern inventions, and the present growth of modern civilization must be attributed largely to the exact sciences and technical arts. These, in turn, are based upon and closely allied with mathematics. As Kent said: "A science is exact only in so far as it employs mathematics." Even though this be true, it would be a great mistake to suppose this practical application may be extended to the great mass of the

people. The percentage of students who are likely to have practical applications of Plane Geometry is certainly very small.

We cannot hope to justify the study of Plane Geometry, then, on the basis of practical application. There must be other and more important reasons for its study.

The proper study of Plane Geometry exercises, or should exercise, the reasoning power *more* and claims from the memory *less* than any other secondary school subject. The man who knows or remembers the most mathematical facts is not the good mathematician, but rather the man who can apply these facts intelligently, who can discover facts that are new to him, and who can reconstruct, by reasoning power, those which he has forgotten.

It is power and not knowledge that offers the true test of mathematical ability. Mathematical power can be developed in its study by the simplicity and gradation of the reasoning processes. The accuracy with which a conclusion may be reached causes accuracy of thinking and a certainty of the result, once it is obtained.

Mathematical reasoning is logical thinking and aids in the development of original thinking. In no subject is this more true than in Plane Geometry.

Since the ultimate aim in the study of Plane Geometry is the acquisition of power and not knowledge, the question naturally arises: What is the best plan of studying in order to reach this goal?

There are those who insist that if, in the study of Plane Geometry, the student memorizes a required number of theorems, can offer a synthetic proof of them and solve a certain number of originals he has mastered the subject. One of the strongest arguments for the traditional procedure of stating a proposition and continuing with the proof is that the plan has wonderful definiteness. That is, one starts from this point and goes to that point and there is no question of where one is going. Quite true, it is definite, yet—

In the study of science, it is commonly accepted that we are endeavoring to develop the scientific attitude of mind for the approach towards the forming of conclusions and setting up of laws governing certain phenomena. The experimental work would precede the actual study of this unit in the text reference. After performing the experi-

ment and studying the situation the student comes to certain conclusions and establishes, for himself, after investigation, that certain reactions follow certain conditions. To that student, Science is not simply a verification of laws that some one has set up, but rather an investigation in which he is the investigator and the conclusions reached are his conclusions that are tested by him to see if they will stand up under varying conditions.

If the student goes into the laboratory to verify the laws or statements learned from the text or reference work, his problem has definiteness about it, but what about the investigative instinct, the scientific attitude we wish to develop, the joy of having discovered certain facts for himself, and the development of his individual reasoning power? Would it be better to teach a student in arithmetic that six oranges at five cents each cost thirty cents or to give the situation to him and let him find out the cost?

In a similar manner, would it not be a wiser thing and develop more reasoning power if a condition is set up and the student is allowed to investigate for himself, follow the truth as he can establish it, and come to the development of a theorem as his final achievement? Permit the student to approach the theorem with the spirit of a discoverer or an investigator and Plane Geometry becomes an adventure.

The latest textbooks being prepared by some authors show a marked trend towards the investigative method. Teachers who are dissatisfied with the results accomplished by the traditional method of study are devising more desirable methods of approach, are leading students to investigate the situation laid down and discover for themselves the truths that lead to the establishment of a theorem. They are growing away from the textbook plan and are becoming more conscious of the fact that we are teaching students to investigate and think for themselves, and that Plane Geometry is one of the greatest tools of secondary school mathematics to develop this power.

A new era is dawning in the study of Plane Geometry. The traditional method must give way to the investigative method. Let us make it investigative rather than verifying, scientific rather than haphazard, an adventure rather than a task to be performed. Let us develop the power of reasoning so that our students shall be able not only to operate better machines but to build better machines.

ON INTUITIVE GEOMETRY

By W. PAUL WEBBER
Louisiana State University

A teaching principle that can hardly be over estimated was laid down by the committee of the M. A. A. and is stated as follows: "The primary purpose of teaching mathematics should be to develop those powers of understanding and of analyzing relations of quantity and space which are necessary to an insight into and a control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and action which make these powers effective in the life of the individual." With this in mind let us consider some phases of elementary geometry.

The formal study of geometry should be preceded by a reasonable amount of informal or intuitive study of space forms. Such a study should consist of observations, experiments, measurements and calculations relative to fundamental geometric forms. Formal geometry was originally intended for adults and for many years or even centuries teachers and text book writers failed to grasp the meaning of this fact. In recent years however, much attention has been given to the subject. The work of intuitive geometry has for one of its purposes the familiarizing of the student with the fundamental geometric forms and the necessary new vocabulary that must attend such work. It is intended further to have the student form the habit of observing geometry in his environment. The outcome will be a considerable stock of useful geometric knowledge and, incidentally, new and undetermined relations will come to the attention of the pupil before he is called upon to follow a strict logical procedure. This will remove one serious stumbling block that worried teachers and students for many years. Just what constitutes a reasonable amount of introductory intuitive geometry may depend on circumstances. No doubt some of this work was touched upon in the grades. It will be necessary in such cases only to recall such work to re-orient it for the study of geometry proper.

The new terms, the figures they are to study about should all be made familiar, so that the difficulty of geometric logic can be attacked with full force. One of the attitudes most needed in all scientific

study is curiosity. That is difficult to obtain in this age of almost unlimited facilities for distracting entertainment. Once a student's curiosity is aroused, the teacher's labor is lightened by half. There will be time here to mention only a few examples of the type of work that can be done, and that should be done, to suitably prepare the student for this attack on formal geometry.

No complete outline of a course in intuitive geometry is attempted here. Only a few suggestions are offered. It is not here recommended that all the material of a course in intuitive geometry be given consecutively. It is rather to be given in small lessons as needed in the approach to new topics of the main course.

(a) Call attention of pupils to geometric forms in their environment. Did they ever see a square tree? Why are houses usually built in rectangular forms? What is the shape of a drop of water or a soap bubble? Are there any triangular forms in the environment? Is there a reason why a bridge truss is formed with triangles? Would quadrilaterals or circles answer as well? Do these questions, and many others that may be asked, suggest a reason for the study of geometry?

(b) Direct measurement of distances by use of tape line or yard stick; measurement of angles by use of the protractor: Measure the dimensions of a room and calculate the area of the floor. Measure the dimensions of a chalk box or similar solid and calculate the volume, also the surface area of the box, same for a tin fruit can by rules given in arithmetic. Are these results or the rules from which they are calculated known by the pupils to be correct? Would it be a matter of interest to be reasonably certain about?

(c) Let us now consider some fundamental facts about triangles. Let the pupils construct triangles in all possible cases from the least number of given parts out of the total of six parts of a triangle, the three sides and the three angles. Answer the questions: Can a definite triangle be constructed when the three sides are given. Make the construction. Will all triangles constructed with the same three sides be exactly alike?

Can a triangle be constructed when its three angles alone are given? Try to construct it. Will all triangles having their angles alone equal be exactly alike? How will they differ if at all?

Take two sides and their included angle as given and construct the triangle. Will all triangles constructed from these same parts be exactly alike?

Take two angles and the side joining the vertices of these two angles as given. Will all triangles constructed from these given parts be exactly alike?

Now take two angles and a side opposite one of these sides and construct the triangle. Construct it in all possible ways. Are all these triangles exactly alike?

Again take two sides and an angle opposite one of these sides and construct the triangle. Are all the possible triangles constructed from these parts exactly alike?

Can you construct a triangle from data given in any different way from the above cases? Which of the above cases make the corresponding triangles exactly alike? These will be proved logically later in your study of geometry.

The above cases of triangles furnish a basis for a number of interesting examples of indirect measurement of distances. (a) Let A and B be 300 feet apart on a level and let a point C be such that as seen from A the angle BAC is 60° and the angle ABC as seen from B is 40°, to find the distance AC and the distance from C to the line AB and the angle at C.

Choose some suitable unit and draw AC to scale. At A and B construct the given corresponding angles. Produce the unknown sides of these angles until they meet. Their intersection determines the position of the point C. By which of the above cases of like triangles do we arrive at this conclusion. Do you now see how you could determine the distance to some far away tower or the width of a river if you had a tape line and an instrument for measuring angles? Would you have to actually go to the tower or actually cross the river to do this?

With these suggestions it is easy to pass to other types of geometric figures, such as isosceles triangles, equilateral triangles and regular polygons. It is insisted that all pupils take part in the actual measurements as well as in the calculations. There is psychological value in

handling a tape line and a protractor that can hardly be had in any other way.

It is needless to say that this program, if taken seriously, places a responsibility on the teacher that can hardly be executed unless that teacher has some training for it and a personal interest in such work.

This raises the age old question of the preparation of teachers. It will be merely asked here if one would go to a veterinary if a child were seriously ill. Would one select an ox to race with a high bred race horse. The answer is obvious. This is still a problem in school administration? No mechanical method, no amount of supervision can substitute successfully for an enthusiastic well prepared teacher.

REFLECTIONS ON THE STUDY OF GEOMETRY AS AN AID TO LOGICAL THINKING

By F. A. RICKEY

Since the appearance of the first edition of the Mathematics News Letter, editorial comment has constantly and forcefully brought out the idea of the "disciplinary" value of the study of mathematics. The belief that there is a definite and valuable "carry over" to other fields of certain analytical and logical modes of thinking has been stoutly defended in the face of a widespread notion that there is no such thing as this kind of value of a subject. Among the high school courses in mathematics, geometry stands out as the subject whose place as a required subject in all standard curricula of our state high schools is due to the belief that it affords training in logical thinking and clear, forceful presentation of argument. Its existence as a required subject cannot be properly based upon its practical uses, value as preparation for other mathematics, or its cultural advantages, important as these objectives may be.

However, the fact that the books of geometry consist largely of a logical sequence of theorems logically proven by no means guarantees training in logical thinking for the tenth grade student. Ability to "size up" a problem or situation, reason out a solution deductively

express a proof, and know when a thing is proven is developed slowly and only when the teacher plans and teaches the course with this in mind. Even the best of geometry teachers sometimes find themselves teaching the facts of geometry as the chief end. Often a conscious or unconscious desire to "make a showing" on standardized tests is the cause of this. Henry Ford was recently quoted as saying that a man is not educated until he learns to think, no matter how many degrees he has. Teachers of geometry will do well to pause occasionally and determine where their teaching is directing itself.

Students are still often hurried into bewildering (to them) deductive proofs in geometry with an inadequate introduction to the fundamental concepts of the subject. We define the axioms as self-evident truths. Yet we find pupils who learn and use the congruent triangle theorems long before they can tell us what "the same thing" is in "things equal to the same thing are equal to each other". The newly adopted text in mathematics for the eighth grades in Louisiana high schools should prove a great help in this respect, introducing many geometric facts and notions intuitively and offering much easy practice material upon them. Regardless of whether the introduction to geometry comes in the eighth grade or the tenth, or both, teachers should make sure that the following important points are met:

1. It should create interest: by pointing out geometric forms in nature; by showing uses of geometry; by arousing a desire to give proofs.
2. It should teach the meaning of necessary terms.
3. It should give practice in measuring with ruler and compasses.
4. It should contain constructions.
5. It should give familiarity with the facts to be proven later.
6. It should give practice in generalizations . . . that is, in the discovery of theorems.
7. It should show the need of proof.
8. It should give meaning to postulates.

At the same time that the importance of a complete and stimulating introduction is being stressed, it might be well to point out that many teachers who lay stress on the informal introduction make it so informal that much of its preparatory value is lost. Students should be made strictly accountable for the facts and concepts made clear and should develop the habit of using exactness and care in forming statements. Furthermore, the line between introduction and more

formal study should not be sharply defined. Let us remember with John Dewey that, "Logical arrangement is the goal and not the point of departure . . . what is important is that the mind should be sensitive to problems and skilled in their solution."

Appeal has been made to the pupil's common sense throughout the introduction, and this common sense should not be insulted by requiring proof of certain so-called theorems in which the pupil can find no need nor challenge for proof. A geometric proof should be to the pupil a very real adventure in the search after and establishing of truth. Certainly no call to adventure is aroused when a pupil is asked to prove a statement that to him is self-evident. In this connection, Professor T. P. Nunn proposes a larger list of postulates to include:

1. Equality of vertical angles.
2. Angle properties of parallel lines.
3. Properties of figures evident from similarity.
4. Properties of figures which can be demonstrated by superposition.

At least, this list gives us food for thought.

The ideal situation in a geometry class is that in which each pupil is led to so investigate the subject of geometry that he is able to share the thrill of discovery and verification. Something of this feeling can be experienced when one reads of the new theory of "floating continents" which holds that at one time there was no Atlantic Ocean, the land surface of the earth being one great continent on one side of the earth. The strain of uneven rotation caused a dividing of land mass, the parts thus formed supposedly floating slowly apart on the molten interior of the earth. The "thrill" comes when one inspects a map of the world (It was in this way that the theory is said to have had its beginning.) and for the first time sees the way in which the Atlantic coast lines of the Old and New Worlds, together with Greenland, fit almost as perfectly as the parts of a jig-saw puzzle. No, one does not have to be the originator of a theory or theorem to feel the spirit of discovery.

But, "What if the training of geometry does not carry over to other subjects or fields?" some may ask. I quote an eloquent reply from David Eugene Smith, who writes:

"If the knowledge of how to arrange a logical proof in geometry can be made of no value to us in other fields in which deductive logic can be applied; if the perfection of geometry does not give us an ideal

of perfection that helps us elsewhere in our intellectual life; if the succinctness of expression in the statement of geometric truth does not set a norm for statements in non-mathematical lines; if the contact with absolute truth does not have its influence upon the souls of us; if the very style of reasoning does not transfer so as to help the jurist, the physician, the salesman, the publicist, and the educator; if the habit of regorous thinking which usually is first begun in demonstrative geometry, is not a valuable habit elsewhere; if a love of beauty cannot be cultivated in geometry so as to carry over to stimulate a love of beauty in architecture—then let us drop geometry from our required subjects."

ON CERTAIN MAXIMAL VALUES

By T. T. HURST
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As a problem of elementary calculus, we may seek the radius of the sphere that will cause a maximum displacement when dropped into a conical wineglass whose depth is *a* and whose generating angle is α.

First, the equation for the volume of the submerged segment of the sphere will be determined. A line AB is rotated about an axis XX′ forming with XX′ the projection ED which we shall call x. From the midpoint M of AB, MN is drawn perpendicular to XX′. Next OM is drawn perpendicular to AB from XX′. We may now take O as the center of a sphere of radius r and passing through the points A and B. The area of the cone frustum generated by AB is $A = 2\Pi\, AB \cdot MN$. In similar triangles ABF and MNO, AB:BF = OM:MN, or, $AB \cdot MN = BF \cdot OM$. Since $BF \cdot OM$ equals $x \cdot OM$, we have $A = 2\Pi\, x \cdot OM$. Next, consider any number of chords to be drawn under arc AB, whose individual projections on XX′ are x_i. ($i = 1, 2, 3, \ldots\ldots n$). The area generated by each chord is $2\Pi r_i x_i$, where r_i is the distance from O to the midpoint of each chord. The area generated by all the chords is, $A' = 2\Pi \sum_{i=1}^{i=n} r_i x_i$. But as n increases, the sum of the chords approaches the arc AB as a limit, r_i approaches r, and the area A′ approaches the area generated by arc AB. Therefore the area of the zone of a sphere is $2\Pi rx$. Consider

next the volume V generated by ABDE. Volume V_1 of the cone BDCE is $(\frac{1}{3})\Pi \cdot x \cdot \overline{BD}^2$ or $(\frac{1}{3})\Pi \cdot x\,(r^2-x^2)$ when we consider E as coincident with O, as is the case with a zone adjacent to a hemi-sphere such as we are dealing with. The volume V_2 generated by ABO $= (\frac{1}{3})$ $r \cdot A = (\frac{1}{3})\, r \cdot 2\Pi rx = (\frac{2}{3})\Pi r^2x$. $V = V_1+V_2=(\frac{1}{3})\Pi xr^2-(\frac{1}{3})\Pi x^3+(\frac{2}{3})$ $\Pi xr^2=\Pi(r^2x-x^3/3)$.

Referring next to the problem of the wineglass, let a = depth of glass, r = radius of sphere, α = generating angle of glass, and x = distance of center of sphere from the top of the glass. We then have, $\sin\alpha = r/(a-x)$ or $x = (a\sin\alpha - r)/\sin\alpha$. We assume this definition since the sphere in question is obviously tangent to the conic surface $dx/dr = -1/\sin\alpha$. By adding a hemisphere to the segment computed above, we have for the submerged portion; $V = \Pi(\frac{2}{3}r^3 + r^2x-x^3/3)$. $dV/dr = \Pi\,(2r^2 + r^2\,dx/dr + 2xr - x^2\,dx/dr)$. We next substitute for dx/dr and set dV/dr equal to zero to seek a maximum. The result is:

$$O=2r^2-r^2/\sin\alpha+2r(a\sin\alpha-r)/\sin\alpha+(a\sin\alpha-r)^2/\sin^3\alpha$$
$$=2r^2\sin^3\alpha-r^2\sin^2\alpha+2ar\sin^3\alpha-2r^2\sin^2\alpha+a^2\sin^2\alpha-2ar\sin\alpha+r^2.$$
$$=r^2(2\sin^3\alpha-3\sin^2\alpha+1)+r(2a\sin^3\alpha-2a\sin\alpha)+a^2\sin^2\alpha.$$

The coefficient of r^2 will be replaced by c. Then

$$O=r^2+r(\sin^2\alpha-1)2a.\sin\alpha/c+a^2\sin^2\alpha/c.$$
$$r=-2a\sin\alpha(\sin^2\alpha-1)/2c\pm(\tfrac{1}{2})\sqrt{(\sin^2\alpha-1)^2.4a^2\sin^2\alpha/c^2-4a^2\sin^2\alpha/c}$$
$$=1/c[a\sin\alpha(1-\sin^2\alpha)\pm a\sin\alpha\sqrt{(\sin^2\alpha-1)^2-c}]$$
$$=1/c\,.a\sin\alpha\,[1-\sin^2\alpha\pm\sqrt{\sin^4\alpha-2\sin^2\alpha+1-2\sin^3\alpha+3\sin^2\alpha-1}]$$
$$=1/c\,.\,a\sin\alpha\,[1-\sin^2\alpha\pm\sqrt{\sin^2\alpha\,(\sin\alpha-1)^2}]$$
$$=1/c\,.\,a\sin\alpha\,(1-\sin^2\alpha+\sin^2\alpha-\sin\alpha)$$
$$=a\sin\alpha\,(1-\sin\alpha)/(2\sin^3\alpha-3\sin^2\alpha+1)$$
$$=a\sin\alpha/(\sin\alpha+1-2\sin^2\alpha)$$
$$=a\sin\alpha/(\sin\alpha+\cos 2\alpha).$$

In order to determine if this value of r does give a maximum, we shall prove that d^2V/dr^2 is a negative quantity when the above value of r is substituted. Referring again to the equation $x = (a\sin\alpha - r)/\sin\alpha$, we have $dx/dr = -1/\sin\alpha = c$. Let $k = \sin\alpha + \cos 2\alpha = \sin\alpha + 1 - 2\sin^2\alpha$, and we have $r=a\sin\alpha/k$ and

$x = (a\sin\alpha - a\sin\alpha/k)/\sin\alpha = a(k-1)/k$.

From $V = \Pi\,(r^2x-x^3/3+\frac{2}{3}r^3)$,

$dV/dr=\Pi\,(r^2c+2rx-x^2c+2r^2)$

and $d^2V/dr^2=\Pi\,(2rc+2rc+2x-2xc^2+4r)$. Let d^2V/dr^2 be replaced by

V''. Then $V''/2\Pi = 2a \sin\alpha/k \;.\; (-1/\sin\alpha) + a(k-1)/k - a(k-1)/k.$
$(1/\sin^2\alpha) + 2a \sin\alpha/k$
and $V''k/2\Pi a = -2 + \sin\alpha - 2\sin^2\alpha - (\sin\alpha - 2\sin^2\alpha)/\sin^2\alpha + 2\sin\alpha.$
$V''k\sin\alpha/2\Pi a = -2\sin\alpha + 3\sin^2\alpha - 2\sin^3\alpha - 1 + 2\sin\alpha$
$= -(1+2\sin^3\alpha) + 3\sin^2\alpha$. In order for V'' to be negative, $1+2\sin^3\alpha$ must be greater than $3\sin^2\alpha$. Let $\sin\alpha = 1-\delta$, where δ is any number between zero and one. We should then have
$1+2\sin^3\alpha > 3\sin^2\alpha$ or $1+2(1-\delta)^3 > 3(1-\delta)^2$ or
$1+2-6\delta+6\delta^2-2\delta^3 > 3-6\delta+3\delta^2$.
Subtracting $3-6\delta+3\delta^2$ from both sides: $3\delta^2-2\delta^3>0$. Since δ is a fraction, the inequality holds and therefore the value of r gives a maximum.

It is interesting to consider situations which result for certain assigned values of x, remembering that a maximum volume is displaced in each case. For the special case when a hemisphere is submerged, $x=0$. For $x = (a\sin\alpha - r)/\sin\alpha = 0$, $r = a\sin\alpha$. Hence $a\sin\alpha = a\sin\alpha/(\sin\alpha + 1 - 2\sin^2\alpha)$, and $1 = \sin\alpha + 1 - 2\sin^2\alpha$, or $2\sin^2\alpha = \sin\alpha$, or $\sin\alpha = \frac{1}{2}$. Therefore $\alpha = 30°$. And so for a generating angle of 30° we find that the hemisphere gives maximum displacement. When the entire sphere is submerged, $x = r$, and $x\sin\alpha = a\sin\alpha - r$, or $r\sin\alpha + r = a\sin\alpha$, or $r = a\sin\alpha/(\sin\alpha+1)$. Referring again to the general value of r, $a\sin\alpha/(\sin\alpha+\cos^2\alpha) = a\sin\alpha/(\sin\alpha+1$. Dividing by $a\sin\alpha$, we have : $\sin\alpha + 1 = \sin\alpha + 1 - 2\sin^2\alpha$, or $2\sin^2\alpha = 0$, or $\alpha = 0$. This means that the maximum sphere is never totally submerged. For α less than 30°, $\sin\alpha = .5 - e$, where e is a fraction less than .5.

$$x = \frac{a[(.5-e)] - a[(.5-e)]/[1+.5-e-2(.5-e)^2]}{.5-e}$$
$= a(.5+1-e-.5+2e-2e^2-1)/(1+.5-e-.5+2e-2e^2)$
$= a(e-2e^2)/(1+e-2e^2) > 0.$
For α greater than 30°, $\sin\alpha = .5+e$, where e is a fraction less than .5.
$x = a(1+e+.5-.5-2e-2e^2-1)/(1+.5+e-.5-2e-2e^2)$
$= a(-e-2e^2)/(1-e-2e^2) < 0.$
These results show that more than a hemisphere is submerged when $\alpha < 30°$ and that the contrary is true when $\alpha > 30°$.

Suggestions regarding certain points in the discussion are due to my instructor Dr. C. D. Smith.

A NOTE ON VECTORS

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In the conventional treatment of vectors, the fundamental formulas are proved by making use of components. This seems to the writer rather artificial. In some cases it is also rather long.

It is the object of this note to give proofs of the fundamental formulas of vector analysis directly on the basis of the definitions and without the use of components. Also, in spite of the fact that there is already a superabundance of notations and terminologies in this subject, a new name and symbol for the so-called "vector product" is proposed.

1. Definitions and notations. We assume that the reader knows what a vector is. We shall denote vectors by capital letters, scalars by small letters. The length of a vector A will be denoted by $|A|$. By the *product* AB of vectors A, B will be meant the so-called "scalar product" defined by the equation

(1) $AB = |A|\ |B| \cos \angle AB,$

where $\angle AB$ denotes the angle between A and B. By the *determinant* A,B of vectors A,B will be meant the so-called "vector product" defined by the equation

$$|A,B| = (|A|\ |B| \sin \angle AB)\ U,$$

where U is a unit vector ($U^2 = 1$) perpendicular to A and to B and so directed that A,B,U are a positively oriented triple. We assume the reader is familiar with the definition of k A, the product of a vector by a scalar, and with the definition of addition.

2. Certain simple formulas. The following follow at once from the definitions above.

(1) $A^2 = |A|^2$ (2) $AB = BA$

(3) $|A,A| = 0$ (4) $|A,B| = -|B,A|$

(5) If $A \perp B$, then $AB = 0$.

(6) If $|A,B|C \neq 0$ and D is any vector, then numbers a, b, c exist uniquely such that

$$D = a\,A + b\,B + c\,C.$$

(7) If $|A,B| \neq 0$, $|A,B|C = 0$, then numbers a,b exist uniquely such that

$$C = a\,A + b\,B.$$

(8) If $|A,B| = 0$, $A \neq 0$, then a number a exists uniquely such that

$$B = a\,A.$$

The following also hold. (For proofs see, Osgood, Advanced Calculus, Chapter XII).

(9) $k(A+B) = k\,A + kB$.
(10) $(A+B)C = AC + BC$.
(11) $|A,B+C| = |A,B| + |A,C|$.

3. The special LaGrange identity. The formula

(12) $|A,B|^2 = A^2B^2 - (AB)^2$

holds. For

$$\begin{aligned} |A,B|^2 &= A^2B^2 \sin^2 \angle AB \\ &= A^2B^2 - (|A|\ |B| \cos \angle AB)^2 \qquad \text{(by (1))}, \end{aligned}$$

which is (12).

4. Expression of a Vector in terms of two Vectors and their determinant. If $|A,B| \neq 0$, so that $|A,B|^2 \neq 0$, then for every vector C numbers p, q, u exist uniquely such that

(13) $C = pA + qB + u|A,B|$.

To determine p and q, multiply (13) by A and by B getting

(14) $(A^2)p + (AB)q = AC$
$(AB)p + (B^2)q = BC$

On solving (14) we get, by (12),

(15) $p = [(AC)(BB) - (AB)(BC)] \,/\, |A,B|^2$,

$q = [(AA)(BC) - (AC)(AB)] \,/\, |A,B|^2$.

To find u, multiply (13) by $|A,B|$ getting by (5),

(16) $u = (|A,B|C) \,/\, |A,B|^2$.

5. The determinant of three vectors. The three numbers $|A,B|C$, $|C,A|B$, $|B,C|\,A$ are numerically equal since the numerical value of each is equal to the volume of the parallelepiped having A,B,C as three

edges meeting at a vertex. But also their algebraic signs are the same. For the three triples ABC,CAB,BCA are similarly oriented and the orientation in each case determines the sign. Thus

(17) $|A,B|C = |C,A|B = |B,C|A.$

This suggests the notation

(18) $|A,B,C| = |A,B|C.$

The symbol $|A,B,C|$ we call the determinant of A,B,C; it has properties analogous to an ordinary determinant. Thus if two of the vectors A,B,C are equal, then $|A,B,C| = 0$. If two of the vectors A,B,C are interchanged, $|A,B,C|$ changes sign. Also the relation

$$|A,B,C_1+C_2| = |A,B,C_1| + |A,B,C_2|$$

holds. Moreover a multiplication rule similar to that of ordinary determinants holds (by(8)).

6. A formula for $|\,|A,B|\,,C|$. We now prove the formula

(19) $|\,|A,B|\,,C| = (AC)B - (BC)A.$

We note first the special case

(20) $|\,|A,B|\,,A| = (A^2)B \qquad (AB = 0).$

The formula (20) follows from the fact that if $AB = 0$, then $|\,|A,B|\,,A|$ is a vector of the form kB $(k>0)$ of length $A^2|B|$.

Now set

$$B_1 = B - (AB/A^2)A.$$

Then

$$AB_1 = 0,$$

and hence

$$|A,B| = |A,\ (AB\,/\,A^2)A + B_1| = |A,B_1|,$$

so that

$$|\,|A,B|\,,A| = |\,|A,B_1|\,,A| = (A^2)B_1. \qquad \text{(by (20))}$$

Hence

(21) $|\,|A,B|\,,A| = (A^2)B - (AB)A.$

From (21),

(22) $\quad | \, |A,B| \, ,B| = (BA)B - (B^2)A.$

We now prove (19) on the hypothesis that $|A,B| \neq 0$. For then we may use (13) getting

$$\begin{aligned} | \, |A,B| \, ,C| &= | \, |A,B| \, ,pA+qB+u \, |A,B| \, | \\ &= p \, | \, |A,B| \, ,A| + q \, | \, |AB| \, ,B| \\ &= [p(A^2)+q(AB)] \, B - [p(BA)+q(B^2)]A, \end{aligned}$$

which by (14) gives (9).

It remains to consider (19) when $|A,B| = 0$. In this case

$$| \, |A,B| \, ,C| = 0.$$

Also assuming $A \neq 0$, we have

$$B = aA.$$

Hence

$$\begin{aligned} &(AC)B - (BC)A \\ &= (AC)(aA) - [(aA)C]A \\ &= a[(AC)A - (AC)A] = 0, \end{aligned}$$

which completes this part of the proof if $A \neq 0$. The result is obvious if $A = 0$.

It follows at once from (19) that

(23) $\quad | \, |AB| \, ,C| \, . \, |A,B| = 0.$

7. The general Lagrange identity. We now prove the general Lagrange identity,

(24) $\quad |A,B| \, |C,D| = \begin{vmatrix} AC, AD \\ BC, BD \end{vmatrix}$

Assume first that $|A,B| \neq 0$. Then we may set, as in section 4,

$C = pA + qB + u|A,B|,$
$D = rA + sB + v|A,B|,$

where p,q,u are given by (15), (16) and r,s,v are obtained from p,q,u respectively by replacing C by D. Hence

$|C,D| = |pA + qB + u|A,B| \, ,rA + sB + v \, |A \, B| \, |,$

or

$$|C,D| = (ps-qr)\ |A,B| + (ru-pv)|\ |A,B|\ ,A| + (su-qv)\ |\ |A,B|\ ,B|.$$

On multiplying this by $|A,B|$ and using (23), we get

$$|A,B|\ |C,D| = (ps-qr)\ |A,B|.$$

Now let d_{ij} be the determinant whose first column is the i-th column of the array

(25) AA,AB,AC,AD
BA,BB,BC,BD

and whose second column is the j-th column of that array. Then, by (15),

$$p = -d_{23} / |A,B|^2, \qquad q = d_{13} / |A,B|^2,$$
$$r = -d_{24} / |A,B|^2, \qquad s = d_{14} / |A,B|^2,$$

so that

$$ps-qr = (-d_{23}d_{14} + d_{13}d_{24}) / (|A,B|^2)^2,$$

and therefore

$$|A,B|\ |C,D| = (d_{13}D_{24} - d_{14}d_{23}) / |A,B|^2.$$

Now let d be the determinant whose first and third rows are the same as the first row of the array (25) and whose second and fourth rows are the same as the second row of (25). Then $d=0$. But also the Laplace expansion of d with reference to its two first rows is

$2(d_{13}d_{24} - d_{14}d_{23} - d_{12}d_{34})$. Hence

$$d_{13}d_{24} - d_{14}d_{23} - d_{12}d_{34} = 0,$$

so that

$$d_{13}d_{24} - d_{14}d_{23} = d_{12}d_{34} = d_{34}\ |A,B|^2.$$

Hence

$$|A,B|\ |C,D| = d,$$

which is (24).

If $|A,B| = 0$, the left member of (24) is zero. But the right member is zero also, since then the rows of the determinant are proportional on account of (8).

8. A formula for $|A,B,C|^2$. We now prove the formula

$$(26)\qquad |A,B,C|^2 = \begin{vmatrix} AA & AB & AC \\ BA & BB & BC \\ CA & CB & CC \end{vmatrix}.$$

Assume first that $|A,B| \neq 0$, $|A,B,C| \neq 0$. Then, by (6), we may write

$$(27)\qquad |A,B| = aA + bB + cC$$

On multiplying (27) by A, by B, and by C, we get

$$(28)\qquad \begin{aligned} (AA)a + (AB)b + (AC)c &= 0 \\ (BA)a + (BB)b + (BC)c &= 0 \\ (CA)a + (CB)b + (CC)c &= |A,B,C|. \end{aligned}$$

If we denote the right number of (26) by d, we get from (28) by Cramer's rule and section 6,

$$(29)\qquad dc = |A,B|^2\ |A,B,C|.$$

But by multiplying (27) by $|A,B|$, we get

$$(30)\qquad |A,B|^2 = c\ |A,B,C|.$$

Substituting (30) into (29), we get

$$(31)\qquad dc = c\ |A,B,C|^2.$$

Since $c \neq 0$ on account of (29), we may divide (31) by c, getting

$$|A,B,C|^2 = d,$$

which is (26).

If $|A,B| \neq 0$, $|A,B,C| = 0$, the left number of (26) is zero. But the right member is zero also. For in this case we may write, by (7),

$$C = aA + bB,$$

so that the third row of d becomes a times the first row plus b times the second row.

If $|A,B| = 0$, the left member of (26) is zero. But the right member is zero also, since in this case, $B = aA$ or $A = bB$, and the first and second rows are proportional.

PROBLEM DEPARTMENT

Edited by
T. A. BICKERSTAFF
University of Mississippi

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Problems for Solution

No. 4. (Corrected) Proposed by the Editor:

Express the square root of $18+2\sqrt{14}+6\sqrt{2}+2\sqrt{63}$ as the sum of three square roots.

No. 8. Proposed by the Editor:

There were five Sundays in February, 1924. Find the next four years in which February will have five Sundays.

No. 9. Proposed by the Editor:

Solve for x:
$\cos[2\sin^{-1}\{\tan(3\cot^{-1}x)\}]=1.$

No. 10. Proposed by the Editor:

The digits of a three-digit number form an arithmetic progression. The digits of 10 less than the number form a harmonic progression. Find the number.

No. 11. Proposed by H. L. Quarles, University, Miss.:

Using compasses only, find the vertices of a square inscribed in a given circle.

(From Newell and Harper: Plane Geometry No. 16 on page 198)